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World-sheet instantons via the Myers effect and $\mathcal{N} = 1^*$ quiver superpotentials

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Abstract: In this note we explore the stringy interpretation of non-perturbative effects in $\mathcal{N} = 1^*$ deformations of the A_{k-1} quiver models. For certain types of deformations we argue that the massive vacua are described by Nk fractional D3-branes at the orbifold polarizing into k concentric 5-brane spheres each carrying fractional brane charge. The polarization of the D3-branes *induces* a polarization of D-instantons into string world-sheets wrapped on the Myers spheres. We show that the superpotentials in these models are indeed generated by these world-sheet instantons. We point out that for certain parameter values the condensates yield the exact superpotential for a relevant deformation of the Klebanov-Witten conifold theory.

Keywords: Solitons Monopoles and Instantons, 1/N Expansion, Brane Dynamics in Gauge Theories, AdS-CFT and dS-CFT Correspondence.

Contents

1. Introduction

The AdS/CFT correspondence in its principal form proposes a duality between large-N (super)conformal theories (SCFT's) in four dimensions and type-IIB superstring theory on $AdS_5 \times X^5$. This duality may be used as a starting point to obtain further dualities between large-N confining gauge theories in four dimensions and IIB superstring theory. In this context much attention has been focussed on relevant perturbations of the $\mathcal{N} = 4$ $\mathrm{SU}(N)$ theory, namely, the so-called $\mathcal{N} = 1^*$ theory [1]–[5] which has vacua where the theory confines and generates a mass gap, exhibiting an extremely rich phase structure.

More recently in [6] the vacuum and phase structure of $\mathcal{N} = 1^*$ perturbations of the A_{k-1} quiver gauge theories ($\mathcal{N} = 2$ SCFT's with gauge group $\mathrm{SU}(N)^k$ and bi-fundamental matter) have been explored in detail revealing even richer infrared dynamics. One of the most remarkable aspects of this class of relevant perturbations of the $\mathcal{N} = 4$ and $\mathcal{N} = 2$

quiver theories is that the holomorphic sector of the resulting $\mathcal{N}=1$ theories can be computed exactly as a function of N and the gauge couplings (or marginal parameters). In particular, the exact superpotential for the mass-deformed $\mathcal{N}=4$ theory was obtained in [5] while the superpotential for the $\mathcal{N}=1^*$ quiver theories was derived in [6]. The aim of this note is to elucidate, from a stringy point of view, the non-perturbative effects that generate these superpotentials and to point out the ingredients of the string duals of the $\mathcal{N}=1^*$ quiver theories. For a certain class of deformations we demonstrate that detailed features of the superpotentials can be understood simply via the Myers effect [3, 7] operating on D-instantons turning them into string world-sheet instantons. The superpotentials are generated by string instantons wrapping flux-supported two-spheres obtained by polarization of D3-branes into 5-branes.

The exact superpotentials and condensates [5, 6] for the $\mathcal{N}=1^*$ theories are elliptic functions of the marginal parameters (the couplings of the individual gauge groups in the quiver) $\tau_i = 4\pi i/g_i^2 + \theta_i/2\pi$ with the IIB string coupling $\tau = i/g_s + C^0$ being the modular parameter. These have a simple semiclassical ($\tau_i \to i\infty$) expansion which in a generic vacuum can be interpreted as contributions from objects carrying a certain topological charge with respect to each gauge group factor of the quiver. In particular for vacua in the Higgs phase these are simply instanton contributions while for vacua in the confining phase the corresponding expansions appear to encode contributions from configurations carrying fractional topological charge within each gauge group factor of the quiver.

In this note we will show how the non-perturbative physics controlling the holomorphic sector of the massive vacua of $\mathcal{N}=1^*$ quiver theories can be understood in a relatively simple way in the IIB theory via D3-branes at an orbifold in the presence of non-trivial fluxes corresponding to the $\mathcal{N}=1^*$ deformations. By analysing appropriate D(-1)-D3-brane configurations at weak string coupling and finite N, we outline "selection rules" that explain the detailed features of instanton contributions that generate the exact superpotentials in the Higgs vacua of $\mathcal{N}=1^*$ theories. By S-duality we obtain the corresponding pictures for the confining vacua of these theories as well.

We focus mainly on $\mathcal{N}=1^*$ perturbations that respect the symmetry under exchanges of the SU(N) factors in the quiver. In particular this implies equal $\mathcal{N}=2$ SUSY preserving masses m for the k bi-fundamental hypermultiplets and equal $\mathcal{N}=1$ SUSY preserving masses μ for the k adjoint chiral multiplets via the following tree-level superpotential perturbation:

$$\Delta W = -\frac{1}{g_{\mathrm{YM}}^2}\sum_{i=1}^{k}\left(m\,\mathrm{Tr}\,[\Phi_{i,i+1}\Phi_{i+1,i}] + \mu\,\mathrm{Tr}\,\Phi_i^2\right). \tag{1.1}$$

The exact low-energy effective superpotential is then,

$$W_{\mathrm{eff}} = -\frac{\mu}{g_{\mathrm{YM}}^2}\sum_{i=1}^{k}\left\langle \mathrm{Tr}\,\Phi_i^2\right\rangle . \tag{1.2}$$

In the Higgs vacua of the resulting A_{k-1} $\mathcal{N}=1^*$ theory we find that the (fractional) D3-branes at an A_{k-1} orbifold polarize into precisely k D5-branes via the dielectric effect [3, 7] by acquiring non-commuting positions. We also argue that the Higgs vacuum configurations suggest that each of these k polarized D5-branes must be homologous to a topologically distinct S^2 cycle of the resolved orbifold. An interesting aspect of the vacuum solutions is that they spontaneously break the symmetry of the theory under exchanges of the $\mathrm{SU}(N)$ factors. From the point of view of the IIB string picture this implies that twisted sector modes are turned on.

We find, subject to certain caveats, that the vacuum equations of the D-instanton gauge theory in the presence of these expanded or polarized D5-branes have solutions that allow only specific numbers of D-instantons to contribute to the expanded brane configuration in a given vacuum. This explains, for instance in the $k = 1$ case, why instantons contribute only in multiples of N in the Higgs vacuum of the $\mathrm{SU}(N)$, mass deformed $\mathcal{N} = 4$ theory. More elaborate selection rules emerge for the A_{k-1} quiver theories in general. Importantly, the polarization of the fractional D3-branes as a response to the background flux, *induces* a polarization of the "dissolved" D-instantons into wrapped D-string world-sheets whose action is exactly computable at weak string coupling. Thus even at arbitrarily weak coupling and finite N, non-perturbative effects in the Higgs vacua can be understood in the IIB D-brane setup as world-sheet instanton contributions. This provides a "microscopic" picture of the world-sheet instantons of [2, 3] which are responsible for non-perturbative effects in the IIB string duals in the large-N limit of $\mathcal{N} = 1^*$ theory at large 't Hooft coupling. S-duality turns the Higgs phase into a confining phase while simultaneously turning the wrapped D-string into a wrapped F-string. In this sense it is natural to think of non-perturbative effects in the confining vacua as being due to F-string world-sheets wrapping NS5-branes obtained by polarizing the fractional D3-branes at the orbifold.[1]

A nontrivial ingredient in the above interpretation is that world-sheet instanton actions wrapping the polarized 5-branes at the orbifolds are expected to be sensitive to twisted sector fields. This is because the polarized 5-brane-spheres are homologous to the two-cycles at the resolved orbifold. Hence wrapped F-string world-sheet actions should be accompanied by phase factors proportional to the integral of the NS or RR two-form over the associated two cycles. In fact precisely such phases appear in the confining vacuum in a strong-coupling expansion of the superpotential via terms proportional to powers of $\exp(i \int B_{NS}/2\pi\alpha')$. Thus, the field theory in the confining vacua appears to naturally encode stringy effects in a strong coupling expansion.

For the sake of simplicity, for the most part we restrict our attention to explicit results in the A_1 or $\mathrm{SU}(N) \times \mathrm{SU}(N)$ quiver model. Our results can be extended straightforwardly to the A_{k-1} models for general k.

We also comment on the exact superpotential for the $\mathcal{N} = 1^*$ deformation of the $\mathrm{SU}(N) \times \mathrm{SU}(N)$ theory where the two adjoint chiral multiplets have masses $\mu_1 = -\mu_2 = \mu$. This theory may be viewed as a relevant deformation of the Klebanov-Witten conifold theory [8]

$$\Delta W = -\frac{1}{g_{\mathrm{YM}}^2} \left(\mu \operatorname{Tr} \Phi_1^2 - \mu \operatorname{Tr} \Phi_2^2 + m \operatorname{Tr} \Phi_{1,2} \Phi_{2,1} + m \operatorname{Tr} \tilde{\Phi}_{2,1} \tilde{\Phi}_{1,2} \right) . \tag{1.3}$$

[1] Strictly speaking the expansion of the superpotentials in terms of world-sheet instantons in the confining vacuum is valid in the regime of large 't Hooft coupling $g_s N \gg 1$ which coincides with the SUGRA regime.

For large hypermultiplet masses $m \gg \mu$, this is softly broken $\mathcal{N}=2$ SUSY Yang-Mills with massive bi-fundamental hypermultiplets and gauge group $\mathrm{SU}(N) \times \mathrm{SU}(N)$. For $m \ll \mu$, however, it can be thought of as a deformation of the Klebanov-Witten theory which in turn may be viewed as a perturbation of the A_1 $\mathcal{N}=2$ quiver theory. Based on the analysis of [6], remarkably, the massive phases and massive vacuum structure and condensates of the resulting theory are identical to that of the theory with tree-level superpotential (1.1). This allows us to evaluate the superpotential in the massive vacua exactly.

2. $\mathcal{N}=1^*$ quiver models: a brief review

The A_{k-1} quiver gauge theories with $\mathcal{N}=2$ supersymmetry describe the low-energy dynamics on the world-volume of N D3-branes at a $\mathbb{Z}_k$ orbifold singularity in an A_{k-1} ALE space [9].

This theory is conformal and has k exactly marginal couplings, one for each gauge group factor

$$\tau_i = \frac{4\pi i}{g_i^2} + \frac{\theta_i}{2\pi}; \qquad i = 1, 2, \ldots k\,. \tag{2.1}$$

We may also naturally define an 'overall' gauge coupling

$$\tau = \sum_{i=1}^{k} \tau_i = \frac{4\pi i}{g_{\mathrm{YM}}^2} + \frac{\theta}{2\pi} \tag{2.2}$$

to be thought of as the gauge coupling for the diagonal $\mathrm{SU}(N)$ factor and is also the coupling constant of the IIB string $\tau = i/g_s + C^0$. In addition to k $\mathcal{N}=2$ vector multiplets (W_α^i, Φ_i), the quiver theory has k bi-fundamental hypermultiplets $\{\Phi_{i,i+1}, \Phi_{i+1,i}\}$ transforming in the $((\mathbf{N}, \bar{\mathbf{N}}), (\bar{\mathbf{N}}, \mathbf{N}))$ representation of the i^{th} and $i+1^{\mathrm{th}}$ $\mathrm{SU}(N)$ factors in the quiver $(i = 1 \ldots k)$. The conformal $\mathcal{N}=2$ quiver theory also has a tree-level superpotential

$$W = \frac{2\sqrt{2}}{g_{\mathrm{YM}}^2} \sum_{i=1}^{k} \mathrm{Tr}\, \Phi_i \left(\Phi_{i,i+1}\Phi_{i+1,i} - \Phi_{i,i-1}\Phi_{i-1,i}\right). \tag{2.3}$$

The $\mathcal{N}=1^*$ quiver theory is obtained by adding to the superpotential (2.3) generic $\mathcal{N}=1$ preserving masses μ_i for the k adjoint chiral multiplets, and generic masses m_i for the bi-fundamental hypermultiplets,

$$\Delta W = -\frac{1}{g_{\mathrm{YM}}^2} \sum_{i=1}^{k} \left(m_i\, \mathrm{Tr}\left(\Phi_{i,i+1}\Phi_{i+1,i}\right) + \mu_i\, \mathrm{Tr}\, \Phi_i^2\right). \tag{2.4}$$

For generic mass parameters, the determination of the vacuum structure is somewhat involved [6], but the final picture that emerges is rather simple and nicely generalizes the story for the mass-deformed $\mathcal{N}=4$ theory. In particular the theory has discrete vacua with mass gap, and

$$\text{No. of massive vacua} = N^{k-1} \sum p\,, \qquad \text{where } p \text{ divides N.} \tag{2.5}$$

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Vacua labelled by a divisor p arise from classical configurations that preserve an $\mathrm{SU}(p)^k$ gauge symmetry with $\mathcal{N}=1$ SUSY. The theory has a $\mathbb{Z}_N$ symmetry that resides in the center of diagonal $\mathrm{SU}(N)$ gauge transformations. Thus as in the case of the mass-deformed $\mathcal{N}=4$ theory, the massive phases of the theory in these vacua are classified by the order N subgroups of $\mathbb{Z}_N \times \mathbb{Z}_N$ [3, 10, 11]. Each massive phase is realized with a multiplicity N^{k-1}. Specifically, there are N^k vacua where the theory is in a (electric) confining phase (with $p=N$) and N^{k-1} vacua where it is in the Higgs phase ($p=1$). As in the case of the mass-deformed $\mathcal{N}=4$ theory, $\mathrm{SL}(2,\mathbb{Z})$ duality on τ of the $\mathcal{N}=1^*$ quiver theory relates the theories in the vacua with different phases, while simultaneously rescaling the individual gauge couplings τ_i. Actually, the conformal $\mathcal{N}=2$ quiver model has an extended duality group [12, 13, 14] which acts by permutation on all the massive vacua of the $\mathcal{N}=1^*$ quiver theory [6].

In this note we will focus attention primarily on the Higgs and confining vacua of the $\mathcal{N}=1^*$ quiver models. Our observations can be carried over with some modifications to include all other vacua and phases of these models.

3. The Higgs vacua in field theory

For the sake of simplicity we will first assume equal bi-fundamental masses $m_i = m$ and equal masses for the adjoints $\mu_i = \mu$. Many qualitative features of the physics that we discuss, will not be affected by this simplifying choice. Furthermore, we will use the vacua in the Higgs phase as our starting point since these can be reliably understood at weak coupling $g_i^2 N/4\pi \ll 1$. The action of dualities on the resulting picture will describe vacua in other (confining) phases.

The classical superpotential of the $\mathcal{N}=1^*$ quiver theory can then be recast as [6]

$$W = \frac{1}{g_{\mathrm{YM}}^2}\,\mathrm{Tr}\left(\Phi\left[\Phi^+,\Phi^-\right] - m\Phi^+\Phi^- - \mu\Phi^2\right) \tag{3.1}$$

where $\Phi^\pm$ and Φ are $kN\times kN$ matrices into which the k bi-fundamentals and adjoints, respectively, can be conveniently amalgamated. We choose an ordering for the elements so that the rows and columns numbered $i, i+k, i+2k, \ldots i+(N-1)k$ are associated with the i^{th} $\mathrm{SU}(N)$ factor in the quiver. With this ordering in place the only non-zero elements of Φ are in the positions $(u, u+nk)$ with n an integer. Similarly the only non-zero elements of $\Phi^\pm$ are in the positions $(u, u\pm 1+nk)$ with n an integer.

With our simplifying choice of mass parameters, the F-term equations are simply the $\mathrm{SU}(2)$-commutation relations after a trivial rescaling of the field variables:

$$\left[\Phi,\Phi^\pm\right] = \pm m\Phi^\pm\,, \qquad \left[\Phi^+,\Phi^-\right] = 2\mu\Phi\,. \tag{3.2}$$

As usual solving D and F flatness conditions and modding out by gauge transformations is equivalent to solving the F-term equations modulo complexified gauge transformations. The complex gauge transformations can be used up to bring the matrices $\Phi, \Phi^\pm$ to a form where $\Phi \propto J_3$ and $\Phi^\pm$ have only off-diagonal non-zero elements in the positions

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$(u, u \pm 1)$ as is to be expected for step-up and step-down generators $J^{\pm}$ of the SU(2) algebra. Massive vacua are associated with special *reducible* representations of the SU(2) algebra.

It was shown in [6] that for solutions to eq. (3.2) which break the gauge group completely, i.e. in the Higgs vacua, the matrices $\Phi, \Phi^{\pm}$ form reducible representations of the SU(2) algebra made up of precisely k irreducible blocks:

$$\left(\Phi, \Phi^{\pm}\right) = \begin{pmatrix} \left(\Phi_1, \Phi_1^{\pm}\right)_{[\ell_1]\times[\ell_1]} & & & \\ & \left(\Phi_2, \Phi_2^{\pm}\right)_{[\ell_2]\times[\ell_2]} & & \\ & & \ddots & \\ & & & \left(\Phi_k, \Phi_k^{\pm}\right)_{[\ell_k]\times[\ell_k]} \end{pmatrix}, \tag{3.3}$$

where $\sum \ell_r = Nk$ and each irreducible block forms a representation of the SU(2) algebra satisfying the usual Casimir relation:

$$\frac{\Phi_r^2}{m^2} + \frac{1}{2\mu m}\left(\Phi_r^+ \Phi_r^- + \Phi_r^- \Phi_r^+\right) = \frac{(\ell_r^2 - 1)}{4}. \tag{3.4}$$

However, not all such reducible representations break the gauge group completely. When $k = 1$ of course, there is only one such solution which is the $N \times N$ irreducible representation of SU(2) corresponding to the single Higgs vacuum of the mass-deformed $\mathcal{N} = 4$ theory. For $k > 1$, the situation is explained in [6]. We simply quote the results below.

3.1 $\mathrm{SU}(N) \times \mathrm{SU}(N)$ theory

In the simplest non-trivial case with $k = 2$ i.e. the $\mathrm{SU}(N)^2$ $\mathcal{N} = 1^*$ theory, Higgs vacuum solutions correspond to reducible representations composed of 2 blocks with dimensions ℓ_1 and ℓ_2 with $\ell_1 + \ell_2 = 2N$ and $\ell_1 = 1, 3, 5, \ldots 2N - 1$, thus yielding N Higgs vacua. It is important to note that based on the ordering of elements chosen above, the odd-numbered rows and columns are associated to the first $\mathrm{SU}(N)$ factor, while the even-numbered ones are associated to the second $\mathrm{SU}(N)$ factor. For general k there are N^{k-1} distinct classical solutions that Higgs the gauge group completely.

3.2 $\mathrm{SU}(N)^k$ theory

For general k the Higgs vacua may be classified as follows [6]. They correspond to partitions of the *ordered* set of kN diagonal elements of $\Phi \propto J_3$ into precisely k sets each of dimension[2] ℓ_k

$$\Big\{1, 2, \ldots, k, 1, 2, \ldots, k, \ldots\ldots, 1, 2 \ldots, k\Big\} \longrightarrow$$
$$\longrightarrow \Big\{ \underbrace{1, \ldots, i_1}_{\mathcal{A}_1} \Big| \underbrace{i_1 + 1, \ldots, i_2}_{\mathcal{A}_2} \Big| \ldots\ldots \Big| \underbrace{i_{n-1} + 1, \ldots, k}_{\mathcal{A}_k} \Big\}, \tag{3.5}$$

[2]Partitions into a larger number of sets gives rise to massless vacua or vacua with a classically unbroken non-Abelian gauge group.

with $i_k = k$ and importantly

$$\Big\{ i_1, i_2, \ldots, i_{k-1} \Big\} = \Big\{ 1, 2, \ldots, k-1 \Big\} . \qquad (3.6)$$

It is important to note that two partitions $\{\mathcal{A}_r\}$ and $\{\mathcal{A}'_r\}$ are equivalent if the second partition yields the first upon a simple re-labelling of the subsets $\mathcal{A}'_r$. They are related by the action of the Weyl group of $\mathrm{SU}(N)^k$.

The number of such distinct partitions can be determined easily. There is a single i_r in (3.5) associated to each of the k gauge group factors. A given i_r can therefore be situated in one of N places. However, i_k is fixed, so the total degeneracy of Higgs vacua is N^{k-1}.

4. Type-IIB picture

The conformal $\mathcal{N} = 2$ quiver theory arises as the low-energy world-volume dynamics of N D3-branes at a $\mathbb{Z}_k$ orbifold singularity in an A_{k-1} ALE space [9]. We choose the D3-brane world-volume to span the coordinates $x^0, \ldots, x^3$. The six-dimensional space transverse to the the D3-branes is $\mathbb{R}^2 \times (\mathbb{R}^4/\mathbb{Z}_k)$. We choose the (x^4, x^5) coordinates to parameterize the transverse plane left fixed by the orbifolding:

$$x^4, x^5 \to x^4, x^5\,; \qquad x^6 + ix^7 \to e^{\frac{2\pi i}{k}} \left(x^6 + ix^7\right)\,; \qquad x^8 + ix^9 \to e^{-\frac{2\pi i}{k}} \left(x^8 + ix^9\right). \quad (4.1)$$

When the D3-branes sit at the orbifold singularity, i.e. on the fixed plane, they are free to fractionate. In general they split into Nk fractional D3-branes which are secretly 5-branes wrapped on one of the $k-1$ collapsed S^2's at the orbifold singularity [15, 16]. The (x^4, x^5) separations between the fractional D3's appear as the k adjoint scalars parameterizing the $(N-1)k$-dimensional Coulomb branch of the quiver model with $\mathrm{SU}(N)^k$ gauge group. Strictly speaking the gauge group is $\mathrm{U}(N)^k$, but the overall $\mathrm{U}(1)$ simply decouples from the dynamics while all the remaining $\mathrm{U}(1)$'s are IR-free and freeze out at low-energies.

The IR-free $\mathrm{U}(1)$ factors correspond to the relative positions of the centers of masses of the k groups of fractional D3-branes on the fixed (x^4, x^5) plane. Non-zero hypermultiplet masses m_i satisfying $\sum m_i = 0$ may be introduced by displacing the centers of masses of these k groups. The $\mathcal{N} = 1^*$ theories have masses μ_i for the k adjoint-valued scalars. As long as the μ_i satisfy $\sum \mu_i = 0$ they may be directly related to blow-up modes of the orbifold singularity [17]. In particular switching on such masses produces resolved spaces which are fibrations of a smooth ALE on the (x^4, x^5) plane.

In the IIB picture, the overall adjoint mass $\sum \mu_i = \mu$ appears when certain components of the 3-form fluxes $F^{(3)} = {}^*dC^{(6)}_{RR}$ and $H^{(3)} = dB^{(2)}_{NS}$ are turned on. The condition $\sum m_i = 0$ can also be relaxed by switching on certain components of the 3-form fluxes so that $\sum m_i = m \neq 0$. This is well understood for the mass-deformed $\mathcal{N} = 4$ theory from the point of view of the AdS/CFT correspondence [3].

4.1 Massive vacua

For the class of deformations discussed in section 3, the IIB setup has only 3-form fluxes switched on. As in the case of the mass deformed $\mathcal{N} = 4$ theory, these fluxes would be expected to polarize the fractional D3-branes at the orbifold.

In particular, with $\mu_i = \mu$ and $m_i = m$, the orbifold singularity remains unresolved, and we find that the collection of Nk fractional D3-branes blows up into k fuzzy spheres (or ellipsoids). This follows directly from the the weak-coupling or classical analysis of the Higgs vacuum configurations where the (non-commuting) positions of the fractional D3 branes satisfy the equations

$$\frac{\Phi_r^2}{m^2} + \frac{1}{2\mu m}(\Phi_r^+\Phi_r^- + \Phi_r^-\Phi_r^+) = \frac{(\ell_r^2 - 1)}{4}\,; \qquad r = 1, 2 \ldots k\,. \tag{4.2}$$

In the Higgs vacuum these fuzzy spheres must be thought of as k concentric D5-branes with world-volumes $\mathbb{R}^4 \times S^2$ with a nontrivial action of the orbifold symmetry on the two-sphere. The dimension ℓ_r of each block $\Phi_r, \Phi_r^\pm$ in the solution eq. (3.3) gives the number of fractional branes constituting the polarized D5-brane described by eq. (4.2). It follows from the classification of Higgs vacua in section 3.2 that each of the k concentric spheres carries a *distinct* fractional D3-brane charge. This is obvious in the $k = 2$ case which we discuss in more detail below. The net 3-brane charge carried by the k concentric spheres is of course N.

In the large N limit, if we look at massive vacua where the dimension ℓ_r of each block is chosen to scale as some positive power of N, the D5-spheres become k smooth spheres embedded in the six-dimensional space transverse to the D3-branes so that two antipodal points intersect the (x^4, x^5) fixed plane and are fixed under the orbifold action. We remark that the Myers spheres we have just described live in the flat orbifold, as our analysis of the Higgs vacuum is classical, i.e. $g_i^2 N/4\pi \ll 1$. The fact that each of the polarized spheres carries a distinct fractional brane charge implies that they must each (non minimally) wrap topologically distinct two-cycles at the orbifold singularity.

The type-IIB configuration for vacua in other phases can be obtained by the action of $\mathrm{SL}(2,\mathbb{Z})$ duality on the above setup. For example the confining vacua would be described by k concentric NS5-branes each carrying a distinct fractional D3-brane charge.

In the large N, $g_s N \gg 1$ limit, we expect the qualitative picture of the concentric D5-brane spheres to survive as in the string dual of the mass-deformed $\mathcal{N} = 4$ theory of Polchinski and Strassler. The string dual of the A_{k-1} conformal quiver theory is the type-IIB theory on $AdS_5 \times S^5/\mathbb{Z}_k$ [18] where the orbifold action leaves fixed a great circle $S^1 \subset S^5$. The $\mathcal{N} = 1^*$ deformation will result in a warped geometry where the k concentric 5-brane spheres will wrap an $S^2 \subset S^5$ intersecting the fixed circle at two antipodal points. In a generic massive vacuum the 5-brane spheres will have different radii and are thus expected to sit at different values of the AdS radial coordinate. Furthermore since each of the 5-branes carries a fractional D3-brane charge, normalizable modes for twisted sector fields corresponding to VEVs in the gauge theory will be turned on.

4.2 $\mathrm{SU}(N) \times \mathrm{SU}(N)$ $\mathcal{N} = 1^*$ quiver theory

We saw in section 3.1 that the $\mathrm{SU}(N)^2$ theory has N Higgs vacua where the gauge group is completely broken. In the basis that we have chosen, the scalar field VEVs in these vacua decompose into 2 irreducible blocks, each forming a representation of the SU(2) algebra. The dimensions ℓ_1 and ℓ_2 of these blocks are odd integers satisfying $\ell_1 + \ell_2 = 2N$.

Thus, in each Higgs vacuum of the $\mathcal{N} = 1^*$, A_1 quiver theory, the $2N$ fractional D3-branes polarize into 2 concentric D5-brane spheres with radii given by the dimensions of the blocks. Importantly, since each D5-sphere is constructed from an odd number (ℓ_i) of fractional D3-branes, it must necessarily carry a fractional D3-brane charge. For example, in the case with $\ell_1 = 1$ and $\ell_2 = 2N - 1$, there is one fractional D3-brane stuck at the origin which can be viewed as a D5-brane wrapping the collapsed S^2-cycle at the orbifold. In addition there is a polarized D5-brane carrying $N - 1$ units of whole D3-brane charge and one unit of fractional D3-brane charge. In the large N limit the smooth D5-sphere must wrap the collapsed S^2 non minimally to account for the fractional D3-brane charge.

Confining vacua are obtained by the action of S-duality on the above picture, turning it into 2 wrapped NS5-branes each carrying the appropriate fractional D3-brane charge.

It is also worth noting that our classical solutions for the Higgs vacua with $\ell_1 \neq \ell_2$, i.e. vacua with unequal sized D5-spheres have non-zero classical VEVs for twisted sector fields of the form $\mathrm{Tr}\,\phi_1^2 - \mathrm{Tr}\,\phi_2^2$ where ϕ_1, ϕ_2 are the adjoint scalars in the two gauge group factors. Vacua which are described by two equal sized concentric 5-brane spheres ($\ell_1 = \ell_2$) do not have twisted sector VEVs at the classical level although in general they may in the quantum theory when the gauge couplings for the two factors are chosen to be different.

In the large-N limit the fuzzy spheres become smooth and are D5-branes wrapped on S^2 cycles supported by flux. From our analysis of fractional branes at the unresolved orbifold it is not entirely clear how the polarized spheres relate to the S^2-cycles associated to the orbifold fixed point. The way to approach this question is to investigate the embedding of the D5-spheres in the resolved orbifold geometry with $\mu_1 \neq \mu_2$. We will not address this issue in this note.

5. Non-perturbative effects in $\mathcal{N} = 1^*$ theory

To understand the origin of non-perturbative effects in $\mathcal{N} = 1^*$ theories from a D-brane point of view we begin by exploring the physics of the Higgs vacua. The resulting picture can then be extended to the confining vacua by S-duality. We expect that the physics of the Higgs vacua can be completely understood in the limit of weak coupling where all gauge couplings are taken to be small. In the semiclassical limit, nontrivial physics in the holomorphic sector (in particular, holomorphic in the gauge couplings) of the theory arises via instantons in each gauge group factor. As usual instantons in the 4D gauge theory can be understood in the D-brane language via D(-1)-branes, or D-instantons, in the presence of D3-branes.

We begin by illustrating how instanton effects in the Higgs vacuum at weak-coupling can be understood in the simplest possible example, namely the SU(N), $\mathcal{N} = 1^*$ theory or the mass-deformed $\mathcal{N} = 4$ theory. M-instanton contributions in this theory can be analyzed via the U(M) gauge theory on the world-volume of M D-instantons in the presence of the N D3-branes.

The world-volume theory of D-instantons in the presence of N coincident D3-branes (in the absence of fluxes leading to the $\mathcal{N} = 1^*$ deformation) is a U(M) gauge theory with 8 supercharges [19]. It consists of 6 adjoint-valued real scalars which can be packaged into

3 complex scalars χ_i $(i = 1, 2, 3)$ parameterizing the 6 coordinates transverse to both the D-instantons and the D3-branes in addition to 4 adjoints a_n corresponding to directions transverse to the instantons but tangent to the world-volumes of the D3-branes. In the presence of the D3-branes only 8 supercharges on the D-instanton gauge theory are preserved due to fundamental strings stretched between the two types of branes giving rise to N fundamental hypermultiplets $w^a_u, \tilde{w}^u_b$, $(u, v = 1, 2, \ldots, N$ and $a, b = 1, 2, \ldots M)$ transforming in the $(M, \bar{M})$ of the $\mathrm{U}(M)$ gauge symmetry and in the $(\bar{N}, N)$ of the $\mathrm{U}(N)$ flavor symmetry of the D-instanton gauge theory. In the absence of $N = 1^*$ deformations the D-instanton gauge theory contains a term in the scalar potential:

$$V_{\mathrm{D}(-1)} \sim \sum_i \bar{w}^u_a (\bar{\chi}_i)^a_b (\chi_i)^b_c w^c_u + \sum_i \tilde{w}^u_a (\chi_i)^a_b (\bar{\chi}_i)^b_c \bar{\tilde{w}}^c_u \,. \tag{5.1}$$

We have not written the full scalar potential since only the terms above and modifications to it will be relevant for the following discussion. We also ignore terms in the D-instanton theory that are suppressed in the $\alpha' \to 0$ limit (for instance, quartic couplings of the χ_i's which are suppressed by two powers of α' relative to the above terms).

Turning on the $\mathcal{N} = 1^*$ deformation for the D3-brane theory has a two-fold effect on the D-instanton gauge theory. Firstly, the $\mathcal{N} = 1^*$ deformation introduces 3-form fluxes in spacetime which appear as masses for the adjoint scalars χ_i. This effect is of the same order as the quartic coupling of the χ_i's, i.e. it is down by two powers of α'. However, there is a second effect that does contribute in the $\alpha' \to 0$ limit. This is an effect *induced* by the noncommutative positions of the D3-branes. The key point is that the D3-brane positions appear as hypermultiplet mass parameters in the D-instanton gauge theory, by a modification of the couplings:

$$\chi_i w \to \chi_i w + w \langle \Phi_i \rangle \; ; \qquad \tilde{w} \chi_i \to \tilde{w} \chi_i + \langle \Phi_i \rangle \, \tilde{w} \,. \tag{5.2}$$

In the Higgs vacuum of the mass-deformed $\mathcal{N} = 4$ theory the Φ_i in eq. (5.2) form an $N \times N$ irreducible representation of the SU(2) algebra. These turn into mass matrices for the hypermultiplets on the D-instanton theory leading to the modified F-term equations:

$$[\chi_i]_{M\times M}\, [w]_{M\times N} = [w]_{M\times N}\, [\Phi_i]_{N\times N} \tag{5.3}$$

and similar equations for $\tilde{w}$.

It is now fairly straightforward to see that these equations restrict the values of M for which $w, \tilde{w}$ are non-vanishing.

When $M = 1$, the χ_i are numbers and eq. (5.3) has no nontrivial solutions since the matrices Φ_i are not simultaneously diagonalizable. Similarly, it is easily seen that for any $M < N$, the eq. (5.3) has no nontrivial solutions unless the Φ_i are simultaneously block diagonalizable which is not possible since they are irreducible $N \times N$ representations of the SU(2) algebra. In fact, nontrivial vacuum solutions exist only when M is an integer multiple of N, $M = Nn$ and χ_i are of the form:

$$[\chi_i]_{Nn\times Nn} = \begin{pmatrix} [\Phi_i]_{N\times N} & & \\ & [\Phi_i]_{N\times N} & \\ & & [\Phi_i]_{N\times N} \end{pmatrix} . \tag{5.4}$$

Thus we find that in the Higgs vacuum of the mass deformed $\mathcal{N}=4$ theory D-instantons can sit on the N D3-branes only in multiples of N. Thus the superpotential of the $\mathcal{N}=1^*$ theory in the Higgs vacuum at weak coupling must naturally have the form

$$W \sim \sum_n c_n \exp\left(2\pi i N n\tau\right). \tag{5.5}$$

Ignoring vacuum-independent additive constants the exact superpotential for the mass-deformed $\mathcal{N}=4$ theory has precisely such an expansion at weak-coupling. This exact superpotential was found in [5] and its value in the Higgs vacuum is given by

$$W \sim \mu m^2 \left[E_2(\tau) - NE_2(N\tau)\right], \tag{5.6}$$

where E_2 is the second Eisenstein series [20]. It turns out that the first term proportional to $E_2(\tau)$ is simply a vacuum-independent additive constant [2, 4] that we ignore for the moment. Both in the weak-coupling or semiclassical regime with $\mathrm{Im}(\tau) \gg 1$, and in the large-N limit with $\mathrm{Im}(N\tau) = N/g_s \gg 1$ which is also the SUGRA regime, this superpotential has an instanton expansion

$$W \sim \mu m^2 \sum_n c_n \exp(2\pi i N n\tau) + (\text{vacuum-independent terms}) \tag{5.7}$$

with $c_n = \sum_{d|n} d$. This is precisely the expansion indicated by our weak-coupling kinematical argument for contributions from D-instantons sitting on and tracking the polarized D3-branes.

We now comment on the vacuum-independent additive constant $E_2(\tau)$ that has a natural weak-coupling expansion in terms of arbitrary powers of instantons. These can be understood as arising from point-like instantons which are separated from the D3-branes. These correspond to solutions to eq. (5.3) that we overlooked. The point is that if $w, \tilde{w}$ vanish there is no constraint on M so that arbitrary numbers of D-instantons can contribute. In this case χ_i are not constrained and the D-instantons are not bound to the D3-branes. Intuitively, they are not sensitive to the positions of the D3-branes, the VEVs, and so their contribution is vacuum independent. However, what is not so clear is why there couldn't be contributions from mixed configurations of regular instantons bound to the D3-branes and point-like instantons. We suspect that their absence is due to extra unlifted fermion zero modes corresponding to the superpartners of the relative separation of the two types of instantons. Integration over the associated Grassmann collective coordinates would then yield nothing.

5.1 World-sheet instantons from D-instantons

A rather interesting outcome of the above picture is that we can now directly attribute non-perturbative effects in the $\mathcal{N}=1^*$ theory to world-sheet instantons (even at weak coupling). This follows from eq. (5.4) which demonstrates that in the Higgs vacuum of the mass deformed $\mathcal{N}=4$ theory, instanton contributions occur in multiples of N, where the associated D-instantons acquire non-commuting positions. In particular, positions of the N D-instantons in each of the n groups track the non-commuting positions of the polarized D3-branes. This implies that each group of N D-instantons has polarized into a higher dimensional object, namely a D-string world-sheet wrapping the fuzzy two-sphere. The fact that this object is a D-string follows from the arguments in [7] according to which, Dp-branes spreading out into a transverse two-sphere via non-commuting positions naturally couple to $p+4$-form fluxes and must be thought of as wrapped D$(p+2)$-branes. The action associated to such a wrapped world-sheet is given simply by the sum of the actions of the constituent D-instantons, namely $\exp(2\pi i N\tau)$. The solution in eq. (5.4) thus describes a D1-string world-sheet wrapped n times around the expanded D5-brane with action $\exp(2\pi i n N\tau)$.

Although our arguments belong in a regime that is *a priori* complementary to the SUGRA regime, the resulting picture coincides with that described in the large-N string dual of Polchinski and Strassler [3]. It is a nontrivial fact that the computation of the wrapped world-sheet action in the string dual as the tension times the proper area of the Myers sphere gives the same answer as the weak-coupling picture described above.

$SL(2,\mathbb{Z})$ duality of the IIB theory (or the $\mathcal{N}=4$ theory) permutes the phases and vacua of the $\mathcal{N}=1^*$ theory and the superpotential terms in any vacuum can be attributed to (p,q)-string instantons wrapping the two-spheres of polarized (p,q) 5-branes. In particular, in the confining vacuum S-duality yields an expansion (for $g_sN \gg 1$) in powers of $\exp(-2\pi i\frac{N}{\tau}) = \exp(-2\pi g_s N)$ which must be associated with F-string instantons wrapping the two-cycle of a polarized NS5-brane. This appears to be the right way to think about non-perturbative effects in the confining vacuum in the regime of large 't Hooft coupling or the SUGRA regime as indicated in [2, 3].

6. Non-perturbative effects in the $\mathcal{N}=1^*$ quiver theory

We now extend the arguments developed in the previous section to the $\mathcal{N}=1^*$ quiver models in a straightforward way and subsequently compare with the associated, exact superpotentials.

As before we will focus on the $SU(N)\times SU(N)$ theory. This theory has N Higgs vacua as explained earlier. Non-perturbative effects in these vacua due to instantons in each gauge group factor can be understood by thinking in terms of a $U(M_1)\times U(M_2)$ quiver gauge theory that lives on the world-volume of $M = M_1 + M_2$ fractional D(-1)-branes at the orbifold in the presence of $2N$ fractional D3-branes. As usual the first group of M_1 fractional D-instantons consists of D1-world-sheets wrapping the collapsed cycle at the orbifold singularity and with action $\exp(2\pi i M_1\tau_1)$. The second group may be thought of as wrapped anti-D1-world-sheets with action $\exp(2\pi i M_2\tau_2) = \exp(2\pi i M_2(\tau-\tau_1))$ where τ is the overall gauge coupling defined in eq. (2.2) and maps onto the coupling of the IIB string.

In the absence of the $\mathcal{N}=1^*$ deformations, the $U(M_1)\times U(M_2)$ gauge theory on the D-instantons is an 8 supercharge theory consisting of two complex adjoint scalars each living in one of the gauge group factors and parameterizing the positions of the D-instantons along the (x^4, x^5) plane. In addition there are bi-fundamentals that parameterize their positions in the orbifold directions. As in the 4D quiver gauge theory, these can be amalgamated into three $(M\times M)$ matrices (where $M = M_1+M_2$), χ and $\chi^{\pm}$ which contain the adjoints and the bi-fundamentals respectively. As in section 3, we will choose an ordering of elements where odd-numbered rows and columns will be associated with the first gauge group factor while even numbered rows and columns will transform under the second factor of the D-instanton theory. Strings stretching between each group of fractional D3-branes and the D-instantons associated with that group give rise to N flavors of fundamental hypermultiplets for each gauge group factor of the D-instanton gauge theory. These can be combined into two matrices $w^a_u, \tilde{w}^v_b$ with $u, v = 1, 2, \ldots 2N$ and $a, b = 1, 2, \ldots M = M_1 + M_2$.

In the Higgs vacua the $2N$ fractional D3-branes polarize into two concentric D5-branes each made up of an odd number ℓ_i of fractional D3-branes with $\ell_1 + \ell_2 = 2N$. The non-commuting VEVs of the fractional D3-branes given by $\Phi, \Phi^\pm$ enter as mass parameters for the fundamental hypermultiplets in the D-instanton theory. The vacuum equations of the D-instanton gauge theory then require:

$$[\chi]_{M\times M}\,[w]_{M\times 2N} = [w]_{M\times 2N}\,[\Phi]_{2N\times 2N} \tag{6.1}$$

We recall that in the Higgs vacua of the $\mathrm{SU}(N)^2$, $\mathcal{N} = 1^*$ theory, the matrices $\Phi, \Phi^\pm$ consist of two irreducible blocks of dimensions ℓ_1 and ℓ_2, each forming a representation of the SU(2) algebra (and hence describing two dielectric D5-branes each carrying a fractional D3-brane charge).

We look for solutions where the D-instantons sit on the polarized D3-branes (i.e. with non-vanishing VEVs for the fundamental hypermultiplets). Following arguments identical to those in the preceding section, we find that non-trivial solutions exist only when the matrices $\chi, \chi^\pm$ are block diagonal and consist of k_1 copies of the ℓ_1-dimensional representation of the SU(2) algebra and k_2 copies of the ℓ_2-dimensional representation, k_1 and k_2 being arbitrary integers.

The instanton contributions from such configurations can be enumerated as follows. The polarized D5-brane with ℓ_1 fractional branes contains $(\ell_1+1)/2$ fractional D3-branes of the first type and $(\ell_1-1)/2$ of the second type. Similarly the second Myers D5-brane contains $(\ell_2-1)/2$ D3-branes of the first kind and $(\ell_2+1)/2$ of the second kind. Hence the net instanton contribution from the above configurations is expected to be proportional to

$$\begin{aligned}
&\exp\left(k_1\left[2\pi i\tau_1\frac{\ell_1+1}{2} + 2\pi i\tau_2\frac{\ell_1-1}{2}\right]\right) \times \exp\left(k_2\left[2\pi i\tau_1\frac{\ell_2-1}{2} + 2\pi i\tau_2\frac{\ell_2+1}{2}\right]\right)\\
&= \exp\left[2\pi i\tau_1(k_1-k_2) + 2\pi i\tau\left[Nk_2 + (k_1-k_2)\frac{\ell_1-1}{2}\right]\right]\ .
\end{aligned} \tag{6.2}$$

Here $\ell_1 = 1, 3, 5, \ldots 2N-1$ sweeps out the N Higgs vacua of the $\mathcal{N} = 1^*$ quiver model. It must be emphasized that these arguments are purely kinematical and it is by no means necessary that all the contributions allowed by this mechanism will actually appear in superpotential terms. Furthermore, as in the case of the mass deformed $\mathcal{N} = 4$ theory, point-like instantons (i.e. those that do not actually sit on the D3-branes and have vanishing hypermultiplet VEVs) can and do contribute alongside the above terms.

What appears to be significant in the context of the above mechanism, is that non-perturbative effects at weak coupling in the Higgs vacua can again be directly thought of as world-sheet instanton effects. This interpretation is due to the fact that solutions

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to eq. (6.1) describe k_1 wrappings of a D1-string world-sheet polarized from $k_1\ell_1$ fractional D(-1)-branes sitting on one polarized D5-brane; and k_2 wrappings of a D1-string world-sheet on the second polarized D5-brane. The D-instantons track the non-commuting positions of the fractional D3-branes on which they sit leading to an induced polarization of D1-strings from fractional D-instantons. Not surprisingly, each of these wrapped D-strings carries a distinct fractional D-instanton charge. In the confining vacua one expects F-string world-sheet instantons to account for the non-perturbative phenomena. We show below that analysis of the exact superpotential in these vacua provides rather compelling evidence for this interpretation.

7. The exact superpotential and wrapped world-sheets

The exact superpotentials for the A_{k-1}, $\mathcal{N}=1^*$ quiver models were derived in [6]. For the sake of simplicity we restrict attention to the $k=2$ case in the Higgs and confining vacua. These superpotentials turn out to be modular functions of the overall gauge coupling τ (also the type-IIB coupling) reflecting $\mathrm{SL}(2,\mathbb{Z})$ duality in τ of the parent conformal quiver theory (accompanied by appropriate re-scalings of the individual gauge couplings) [14]. This $\mathrm{SL}(2,\mathbb{Z})$ duality acts on the vacua of the mass-deformed theories by permutation thus exchanging the phases of these theories. In particular S-duality on the Higgs vacua yields confining vacua. This structure is encoded in the condensates in each vacuum and in particular, in the value of the superpotential in a given vacuum. These features are more or less identical to those found in the context of the mass deformed $\mathcal{N}=4$ theory [3, 5, 2, 11].

In addition to $\mathrm{SL}(2,\mathbb{Z})$, the $\mathcal{N}=2$ conformal quiver theory has extra duality symmetries [6, 12, 14]. Part of this duality symmetry is easy to understand from the gauge theory viewpoint as it simply corresponds to shifting the theta angles of one of the gauge groups by multiples of 2π, $\tau_i \to \tau_i + n$. The additional symmetry is visible from the associated string theory setups. In the IIA/M-theory construction of the elliptic models of [12] this duality symmetry can be understood as translations of the NS5-branes (as explained in [6]) by periods of the spacetime torus which leaves the intersecting brane setup unchanged. Periodic motion along one of the cycles (the M-dimension) corresponds to shifts of individual theta angles, while motion along the other cycle of the spacetime torus leads to shifts in the gauge couplings $\frac{1}{g_i^2}$.

In the IIB picture of D3-branes at the orbifold, the gauge couplings are related to the IIB NS and $R-R$ two-form potentials as,

$$\tau_1 = \frac{1}{4\pi^2\alpha'}\left(\tau\int_{S^2} B_{NS} + \int_{S^2} C^2_{RR}\right); \qquad \tau_1+\tau_2=\tau=\frac{i}{g_s}+C^0_{RR}\,, \tag{7.1}$$

where the integrals are performed over the collapsed S^2 at the orbifold. The integrals $\int C^2_{RR}/2\pi\alpha'$ and $\int B_{NS}/2\pi\alpha'$ are both periodic variables with period 2π corresponding to the periodicity of the theta angle and of the Yang-Mills gauge coupling $\frac{1}{g_i^2}$ itself. Upon switching on the $\mathcal{N}=1^*$ deformation these symmetries of the parent conformal theory act on the vacua of the mass deformed theory by permutation. In particular, by performing shifts of the gauge coupling $\frac{4\pi}{g_1^2} \to \frac{4\pi}{g_1^2}+\frac{n}{g_s}$ or $\tau_1 \to \tau_1 + n\tau$ we can clock around the

N Higgs vacua of the $\mathcal{N} = 1^*$ quiver theory. This structure is precisely encoded in the exact superpotentials which were obtained in [6], through the appearance of terms that are elliptic in the individual gauge couplings.

We now quote the results of [6] for the superpotential and condensates evaluated in the Higgs and confining vacua with, for simplicity, equal hypermultiplet masses $m_1 = m_2 = m$. For a general $\mathcal{N} = 1^*$ deformation the superpotential is simply

$$W_{\text{eff}} = -\frac{1}{g^2_{\text{YM}}} \left(\mu_1 \left\langle \text{Tr}\, \Phi_1^2 \right\rangle + \mu_2 \left\langle \text{Tr}\, \Phi_2^2 \right\rangle \right) . \tag{7.2}$$

It is convenient to express this as a linear combination of the condensates

$$\begin{aligned} H_e &= \left\langle \text{Tr}\, \Phi_1^2 \right\rangle + \left\langle \text{Tr}\, \Phi_2^2 \right\rangle , \\ H_o &= \left\langle \text{Tr}\, \Phi_1^2 \right\rangle - \left\langle \text{Tr}\, \Phi_2^2 \right\rangle \end{aligned} \tag{7.3}$$

which are even and odd, respectively, under the exchange of the two SU(N) factors.

These condensates can be expressed in terms of elliptic Weierstrass functions (see Appendix and [21] for details)

$$\begin{aligned} \wp(z) &= -\zeta'(z) , \\ Q(z) &= \zeta(z) - \frac{\zeta(\omega_1) z}{\omega_1} . \end{aligned} \tag{7.4}$$

with half-periods

$$\omega_1 = i\pi; \qquad \omega_2 = i\pi\tau . \tag{7.5}$$

Here we have introduced the function $Q(z)$ which has the property that it is quasi-elliptic rather than elliptic. The linear piece precisely gets rid of an identical term appearing in a semiclassical $\text{Im}(\tau) \gg 1$ expansion of $\zeta(z)$. In terms of these functions the condensates determining the superpotential in the Higgs vacua $s = 0, 1, \dots, N-1$ are (up to vacuum-independent additive constants)

$$\begin{aligned} H_e(s)\Big|_{\text{Higgs}} = &\left\{ \frac{-N^2}{2} m^2 \Big[E_2(\tau) - N\, E_2(N\tau) \Big] + \frac{1}{2Nm^2} H_o(s)^2 - 2s^2 N m^2 + \right. \\ &\left. + 2N^2 m^2 \Big[\wp(2i\pi\tau_1|\omega_1,\omega_2) - N\wp(2i\pi\tau_1 + 2s\omega_2|\omega_1, N\omega_2) \Big] \right\} , \end{aligned} \tag{7.6a}$$

$$H_o(s)\Big|_{\text{Higgs}} = 2Nm^2 \Big[NQ(2i\pi\tau_1 + 2s\omega_2|\omega_1, N\omega_2) - Q(2i\pi\tau_1|\omega_1,\omega_2) + s \Big] . \tag{7.6b}$$

Note that the (quasi-)elliptic functions that encode the vacuum dependence i.e. the s-dependence are (quasi-)elliptic on a torus with complex structure $N\tau$. It is also important to note that both the condensates, as functions of $2\pi i\tau_1$, are actually elliptic (and not quasi-elliptic) on this torus with complex structure parameter $N\tau$. This property can be understood from the viewpoint of the IIA/M-theory setup corresponding to the massive vacua, wherein the M5-brane is actually wrapped on a torus which is an N-fold cover of the spacetime torus with complex structure τ. This leads to a low energy duality, namely $\tilde{S}$-duality [4, 6].

The index s that labels the N Higgs vacua is directly related to the dimensions ℓ_i of the two irreducible blocks characterizing the classical solutions for the Higgs vacuum in eq. (3.3). In particular $s = (\ell_1 - 1)/2$. Thus s is also related to the number of fractional D3-branes ℓ_1, ℓ_2 making up each of the two D5-spheres in the IIB picture of the Higgs vacuum. For instance, $s = 0$ is the Higgs vacuum where precisely one fractional D3-brane remains stuck at the origin, while the remaining $2N - 1$ fractional D3-branes blow up into a dielectric D5-brane.

At weak coupling $g^2_{\rm YM}/4\pi = g_s \ll 1$ and $g^2_i/4\pi \ll 1$, the condensates H_o and H_e in (7.6a) and (7.6b), and hence the $\mathcal{N} = 1^*$ superpotential have an expansion in terms of instantons in each gauge group factor and whole instantons. This expansion is also valid in the large N limit in the SUGRA regime with $\mathrm{Im}(N\tau) = 4\pi N/g^2_{\rm YM} = N/g_s \gg 1$. Using standard formulae for the expansion of Weierstrass $\wp$-functions and the Weierstrass ζ-functions [21], and ignoring the vacuum independent additive contributions we find that the condensates H_o and H_e have a semiclassical expansion of the form,

$$H_{o,e}(s)\Big|_{\rm Higgs} \sim \sum_{k_1,k_2} A_{k_1,k_2} \exp\left(2\pi i k_1 \left[s\tau + \tau_1\right]\right) \exp\left(2\pi i k_2 \left[(N-s)\tau - \tau_1\right]\right). \tag{7.7}$$

The coefficients A_{k_1,k_2} can be easily determined using standard expansions and are completely vacuum-independent constants.[3] Note terms of this form are in complete agreement with the contributions argued in the previous section (see eq. (6.2)) once we identify s with $(\ell_1 - 1)/2$. Each of the terms in the above expansion has the interpretation of D1-string world-sheets multiply wrapping the two polarized D5-spheres, k_1 and k_2 times respectively.

For every Higgs vacuum associated to a given s there exists an S-dual confining vacuum. Thus there are N confining vacua that are S-dual to the N Higgs vacua. These confining vacua are related to each other by shifts of the theta angle of one of the gauge couplings $\tau_1 \to \tau_1 + 1$. Each of these confining vacua is also a member of a family of N vacua related to each other by the shifts $\tau \to \tau + 1$. Hence there are N^2 confining vacua. The S-dual of the semiclassical instanton expansion in the Higgs vacuum which is also a D-string world-sheet instanton expansion then naturally yields an F-string world-sheet instanton expansion in the confining vacua. It is important to note however that the latter expansion in terms of F-string instantons is valid only in the regime of large 't Hooft coupling which is the SUGRA limit.

Without-loss-of-generality we now turn attention to the confining vacuum S-dual to the $s = 0$ Higgs vacuum. The condensates in this vacuum have the form,

$$\begin{aligned} H_e\Big|_{\rm conf.} = \Bigg\{ & \frac{-N^2}{2} m^2 \Big[E_2(\tau) - \frac{1}{N}\, E_2(\tau/N)\Big] + \frac{1}{2Nm^2} H_o^2 \\ & + 2N^2 m^2 \Big[\wp(2i\pi\tau_1|\omega_1,\omega_2) - N\wp(2i\pi\tau_1|N\omega_1,\omega_2)\Big]\Bigg\}, \end{aligned} \tag{7.8a}$$

$$H_o\Big|_{\rm conf.} = 2Nm^2 \Big[NQ(2i\pi\tau_1|N\omega_1,\omega_2) - Q(2i\pi\tau_1|\omega_1,\omega_2)\Big]. \tag{7.8b}$$

[3] These coefficients are simple integers, except for those originating from the term $\propto H_o(s)^2$ in eq. (7.6a). The latter term can be interpreted as arising from an operator mixing with vacuum-independent coefficients that are nontrivial functions of the gauge couplings (see [4, 6] for more comments on operator mixings).

Not surprisingly, the superpotential in the confining vacuum cannot be interpreted in terms of smooth semiclassical configurations such as instantons. This is because in the semiclassical limit the terms in the superpotentials appear to have contributions with fractional topological charge, often loosely termed as "fractional instantons" (these are of course, distinct from fractional D-instantons at the orbifold which correspond to objects with integer topological charge residing in a given gauge group factor). In fact in the limit $\mathrm{Im}(\tau) \gg 1$ and $\mathrm{Im}(\tau_1) \gg 1$, eq. (7.8a) has a natural expansion in terms of powers fractional instantons in each gauge group factor

$$W_{\text{conf.}} \sim \sum_{n,m=0}^{\infty} B_{n,m} e^{2\pi i n\tau_1/N} e^{2\pi i m\tau_2/N} \,. \tag{7.9}$$

However, when we look at the confining vacuum in a SUGRA type limit, i.e. with $g_s N \gg 1$ a very different story emerges. The key point, which is very similar to that outlined in [5, 4], is that the superpotential in the confining vacuum is actually a modular function of the variable τ/N. This is manifest in the term proportional to $E_2(\tau/N)$. However this is also true for the terms involving the elliptic functions. The latter are actually periodic on the torus with half-periods $N\omega_1 = i\pi N$ and $\omega_2 = i\pi\tau$ with complex structure parameter $\tilde{\tau} = \tau/N$. Once again this is a consequence of the fact that holomorphic sector of the theory possesses a low-energy duality symmetry which requires invariance under modular transformations acting on $\tilde{\tau}$ – namely, $\tilde{\mathrm{S}}$-duality [4, 6].

In the semiclassical limit $\mathrm{Im}(\tau/N) = 1/g_s N \gg 1$ the functions involved have the natural expansion as above. However in the strong coupling limit $g_s N \gg 1$ (setting $C^0 = 0$ for simplicity) it is natural to perform the modular transformation $\tau/N \to -N/\tau = i g_s N$ and re-expand in powers of $\exp(-2\pi i N/\tau)$. Under this transformation:

$$E_2(\tau/N) \quad \longrightarrow \quad \frac{N^2}{\tau^2} E_2(-N/\tau) - \frac{6N}{i\pi\tau} \,, \tag{7.10}$$

$$\wp(2\pi i\tau_1 | N\omega_1, \omega_2) \quad \longrightarrow \quad \frac{N^2}{\tau^2} \wp\left(\frac{2\pi i N\tau_1}{\tau} | N\omega_1, -N^2\omega_1/\tau\right) , \tag{7.11}$$

$$\zeta(2\pi i\tau_1 | N\omega_1, \omega_2) \quad \longrightarrow \quad \frac{N}{\tau} \zeta\left(\frac{2\pi i N\tau_1}{\tau} | N\omega_1, -N^2\omega_1/\tau\right) . \tag{7.12}$$

Setting $C^0 = 0$, in the large $g_s N$ limit, $E_2(-N/\tau) = E_2(i g_s N)$ has an expansion in powers of $\exp(-2\pi g_s N)$. Such a term can clearly be obtained by S-duality on a configuration in the $s = (\ell_1 - 1)/2 = 0$ Higgs vacuum where each D5-sphere is wrapped once by a D1-string world-sheet yielding an action proportional to $\exp(-2\pi N/g_s)$ using eq. (6.2).

However the condensates in the confining vacuum encode an additional important piece of information relevant to wrapped world-sheets at the orbifold. Choosing all the theta angles to be zero for simplicity we may rewrite eq. (7.12) for instance as

$$\wp\left(\frac{2\pi i N\tau_1}{\tau} | N\omega_1, -N^2\omega_1/\tau\right) = \wp\left(iN\frac{1}{2\pi\alpha'}\int_{S^2} B_{NS} | N i\pi, -N^2\pi g_s\right) . \tag{7.13}$$

In the regime of large 't Hooft coupling $g_s N \gg 1$, this function can be expanded in powers of $\exp(-2\pi g_s N) \times \exp(in \int B_{NS}/2\pi\alpha')$. The appearance of the phase factors in this expansion is strongly suggestive of the fact that this is indeed an expansion in powers of the F-string world-sheet instantons wrapping the NS5-brane spheres at the orbifold in the confining vacuum. Since the 5-branes themselves are topologically (non-minimally) wrapped around the vanishing S^2 at the orbifold, a string world-sheet wrapping such a 5-brane necessarily picks up a phase contribution from the twisted sector field, namely the integral of B_{NS} over that two-cycle. In particular the superpotential for the $s = 0$ confining vacuum has the form

$$W_{\rm conf.} \sim \sum_{k_2,k_2} A_{k_1,k_2} \exp\left(-2N\pi k_2 g_s + i\frac{(k_1-k_2)}{2\pi\alpha'}\int B_{NS}\right) \tag{7.14}$$

Clearly these terms are consistent with a picture of k_1 F-string world-sheets wrapping the first 5-brane sphere and k_2 wrappings of the second 5-brane. The orientations of the wrappings are opposite since each of the 5-branes carries a different fractional D3-brane charge. This accounts for the fact that the there are phase contributions proportional to $(k_1-k_2)\int B_{NS}$. This expansion of the superpotential terms also implies that the polarized 5-branes must be classified by the homology of the vanishing two-cycles of the orbifold, i.e. they must wrap (non-minimally) the S^2-cycles at the orbifold. We also remark that this world-sheet expansion of the superpotential in the confining vacuum follows directly from an S-duality on the expression in the Higgs vacuum. However it is important to note that the expansion in terms of world-sheet instantons in the confining vacuum is only valid in the regime of large 't Hooft coupling or the SUGRA regime. At weak coupling it only appears to have an expansion in terms of fractional instantons in each gauge group factor, which then get re-summed into world-sheet instantons in the SUGRA limit.

8. Relevant deformation of the conifold theory

One of the remarkable features of the analysis of the $\mathcal{N}=1^*$ quiver theories is that the massive vacuum and phase structure is independent of the actual values of the mass parameters in the deformation. In particular the condensates of chiral operators (in the $\mathcal{N}=1$ sense) are independent of the specific choice of mass parameters. One way to understand this is to view these theories as softly broken $\mathcal{N}=2^*$ theories. From this viewpoint, the massive vacua of the $\mathcal{N}=1^*$ theories are simply the maximally singular points [6] on the Coulomb branch of the unperturbed $\mathcal{N}=2^*$ theories and the condensates of chiral operators are given by the locations of these special singular points on the Coulomb branch moduli space. The latter are of course independent of the choice of $\mathcal{N}=1^*$ parameters. The low energy effective superpotentials for a wide class of $\mathcal{N}=1^*$ theories can thus be automatically determined as they are simply special linear combinations of the chiral condensates.

Thus we may imagine a special $\mathcal{N}=1^*$ deformation of the $\mathrm{SU}(N)\times\mathrm{SU}(N)$ quiver theory where we give masses $\mu_1=-\mu_2=\mu$ to the two adjoint chiral multiplets:

$$\Delta W = -\frac{1}{g^2_{\rm YM}}\left(\mu\,\mathrm{Tr}\,\Phi_1^2 - \mu\,\mathrm{Tr}\,\Phi_2^2 + m\,\mathrm{Tr}\,\Phi_{1,2}\Phi_{2,1} + m\,\mathrm{Tr}\,\tilde{\Phi}_{2,1}\tilde{\Phi}_{1,2}\right). \tag{8.1}$$

When $\mu \ll m$ this theory can be viewed as a soft breaking of $\mathcal{N}=2$ SUSY Yang-Mills with massive bifundamental hypermultiplets. However, when $\mu \gg m$ we may also think of this theory as a relevant deformation of the Klebanov-Witten conifold theory [8]. (With $m=0$, the bifundamental masses correspond to operators with dimension 3/2 after accounting for the anomalous dimensions at the infrared fixed point of the conifold theory.) The low energy effective superpotential for this theory is simply

$$W_{\text{eff}} = -\frac{\mu}{g^2_{\text{YM}}} \left(\langle \text{Tr}\, \Phi_1^2 \rangle - \langle \text{Tr}\, \Phi_2^2 \rangle \right) . \tag{8.2}$$

Recall that in the $m = 0$ theory, one may integrate out the Φ_i to obtain the quartic superpotential of the conifold theory. Thus the effective superpotential must be odd under an exchange of the two $\text{SU}(N)$ factors. Using this and the appropriate semiclassical limit, one finds from the results of [6],[4] the following expression for the exact superpotential of the theory:

$$W_{\text{eff}} \propto H_o\,. \tag{8.3}$$

This is the exact low energy effective superpotential for the deformation of the conifold SCFT by relevant operators corresponding to mass terms for the bi-fundamentals of the orbifold theory.

Note that this superpotential has a semiclassical instanton expansion in the Higgs vacua which appears to satisfy the basic selection rules described earlier. However, the classical vacuum equations can no longer be interpreted in terms of polarization of D3-branes into 5-branes. Instead, 5-branes automatically appear in the IIB picture since the mass perturbation corresponds to a resolved orbifold with blown up S^2 cycles and the fractional D3-branes are D5-branes wrapped on this S^2. Non-perturbative effects in the gauge theory are now simply world-sheet instantons wrapping this blown-up S^2.

9. Conclusions

In this note we have discussed aspects of the type-IIB brane setup of the A_{k-1}, $\mathcal{N} = 1^*$ quiver models with gauge group $\text{SU}(N)^k$ and non-perturbative physics therein. These theories, obtained as relevant deformations of the UV-finite $\mathcal{N} = 2$ quiver theories, have vacua with a mass gap where the theory is realized in Higgs and confining phases. We argue that the IIB setup for these theories involves k polarized 5-branes obtained from Nk fractional D3-branes at the unresolved orbifold in the presence of 3-form fluxes corresponding to the mass deformation. The Higgs vacua are described by k concentric D5-branes with world-volume $\mathbb{R}^4 \times S^2$, while the confining vacua are described by k concentric NS5-branes. In addition each of these 5-brane spheres has a distinct fractional brane charge resulting in the interpretation that they must each be homologous to a distinct S^2-cycle of the resolved orbifold. The non-perturbative effects in the Higgs vacua can be analysed at weak coupling and are found to be due to D-instantons polarized to Euclidean D-string world-sheets wrapping the polarized D5-spheres. This is obtained by analysing the associated D-instanton gauge theory. The latter yield selection rules for the from of instanton contributions in the

[4] In [6] the condensates were directly related to the Hamiltonians of the spin generalisation of the Calogero-Moser integrable system. In this context the Hamiltonian associated with the H_o has precisely the right symmetry property under the exchange and is hence identified with the expectation value of $\langle \text{Tr}\, \Phi_1^2 \rangle - \langle \text{Tr}\, \Phi_2^2 \rangle$.

Higgs vacua. The physics of the confining vacuum is S-dual to this and originates from wrapped F-strings. The exact superpotentials provide a nontrivial confirmation of this picture. The strong coupling expansion of the superpotential produces precisely the terms that would be obtained by wrapping F-strings on the NS5-brane spheres at the orbifold. These constitute the basic ingredients of the large-N string dual of the $\mathcal{N} = 1^*$ quiver theories.

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A. ??? Which title? ???

In this Appendix we state some properties of the elliptic and modular functions in the paper. For further details see [6, 21]. The Weierstrass $\wp$-function is an even function with periods $2\omega_1 \equiv 2i\pi$ and $2\omega_2 \equiv 2i\pi\tau$ defined via:

$$\wp(z) = \frac{1}{z^2} + \sum_{m,n\neq(0,0)} \left\{ \frac{1}{(z - 2m\omega_1 - 2n\omega_2)^2} - \frac{1}{(2m\omega_1 + 2n\omega_2)^2} \right\}. \tag{A.1}$$

The Weierstrass ζ-function is defined via

$$\wp(z) = -\zeta'(z)\,. \tag{A.2}$$

It is quasi-elliptic:

$$\zeta(z + 2\omega_{1,2}) = \zeta(z) + 2\zeta(\omega_{1,2})\,. \tag{A.3}$$

with

$$\omega_2\zeta(\omega_1) - \omega_1\zeta(\omega_2) = \frac{i\pi}{2}\,. \tag{A.4}$$

The function $Q(z) = \zeta(z) - \zeta(\omega_1)z/\omega_1$ satisfies

$$Q(z + 2\omega_1) = Q(z)\,; \qquad Q(z + 2\omega_2) = Q(z) - 1\,. \tag{A.5}$$

Defining $\tau \equiv \omega_2/\omega_1$ and $q \equiv e^{2\pi i\tau}$, $\wp(z)$ has the following (instanton) expansion:

$$\wp(z) = -\frac{\zeta(\omega_1)}{\omega_1} - \frac{1}{4}\mathrm{cosec}^2\left(\frac{\pi z}{2\omega_1}\right) + 2\sum_{n=1}^{\infty} \frac{nq^n}{1 - q^n} \cos\left(\frac{n\pi z}{\omega_1}\right). \tag{A.6}$$

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